

# Path-integral over non-linearly realized groups and hierarchy solutions

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**ABSTRACT:** The technical problem of deriving the full Green functions of the elementary pion fields of the nonlinear sigma model in terms of ancestor amplitudes involving only the flat connection and the nonlinear sigma model constraint is a very complex task. In this paper we solve this problem by integrating, order by order in the perturbative loop expansion, the local functional equation derived from the invariance of the SU(2) Haar measure under local left multiplication. This yields the perturbative definition of the path-integral over the non-linearly realized SU(2) group.

**KEYWORDS:** Renormalization Regularization and Renormalons, Sigma Models.

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## 1. Introduction

The perturbative quantization of the nonlinear sigma model in  $D = 4$  requires a strategy for the definition of the path-integral over the Haar measure of non-linearly realized groups.

It has been recently pointed out [1]- [3] that such a definition can be implemented through the local functional equation which expresses the invariance of the Haar measure under local left group multiplication. The subtraction procedure is required to be symmetric, thus preserving the validity of the local functional equation to all orders in the loop expansion [3].

The local functional equation fixes the Green functions of the quantized pion fields parameterizing the  $SU(2)$  group element (over which the path-integral is performed) in terms of those of the  $SU(2)$  flat connection and the order parameter (ancestor composite operators). This goes under the name of hierarchy principle [1]. Moreover there is only a finite number of divergent ancestor amplitudes at every loop order (weak power-counting theorem [2, 3]).

The local solutions of the linearized functional equation (relevant for the classification of the allowed finite renormalizations order by order in the loop expansion) were obtained in [2]. In the one-loop approximation these results have been shown [2] to reproduce those of ref. [4].

In this paper we show how to explicitly solve the local functional equation by reconstructing the full Green functions of the quantized fields once the relevant ancestor amplitudes are known, to every order in the loop expansion.

In the one-loop approximation (linearized functional equation) this is achieved by group-theoretical methods allowing to introduce a suitable set of invariant variables in one-to-one correspondence with the external sources  $J_{a\mu}$  (coupled in the classical action to the flat connection) and  $K_0$  (coupled to the order parameter). These invariant variables give rise to the whole dependence of the one-loop vertex functional on the quantized fields.

As a special case one can apply this algorithm to the space of local functionals. We then show that the results of ref. [2] are recovered.

At higher orders one has to solve an inhomogeneous equation. For that purpose we make use of algebraic BRST techniques originally developed in the context of gauge theories [5]–[9] in order to invert the linearized operator in the relevant sector at ghost number one.

The main result is that starting from two loops on the dependence of the vertex functional on the quantized fields  $\phi_a$  is two-fold: the  $n$ -th loop ancestor amplitudes induce the dependence on the  $\phi$ 's through the invariant variables solution of the linearized functional equation (implicit dependence). The lower-order contributions (giving rise to the inhomogeneous term as a consequence of the bilinearity of the functional equation) account for the explicit dependence of the  $n$ -th order vertex functional on the quantized fields.

We stress that in this approach the functional equation is recursively solved order by order in the loop expansion. This allows to obtain the full dependence of the vertex functional on the quantized fields (which is uniquely determined once the ancestor amplitudes are known) to all loop orders.

This algorithm can be applied to many problems arising in the quantization of non-renormalizable theories based on the hierarchy principle. We just mention two of them here. The technique discussed in this paper can be applied to higher loops Chiral Perturbation theory [10] in order to determine the full dependence of the vertex functional on the pion fields (including those terms which are on-shell vanishing).

Moreover this method is expected to provide a very useful tool in the program of the consistent quantization of the Stueckelberg model [11]–[13] for massive non-abelian gauge bosons.

The paper is organized as follows. In section 2 we briefly review the subtraction procedure based on the hierarchy principle in the flat connection formalism. In section 3 we solve the local functional equation in the one-loop approximation in full generality. We do not impose any locality restrictions on the space of the solutions. In section 4 we discuss some one-loop examples. We show that by applying the algorithm of section 3 to the space of local functionals the results of ref. [2] are recovered. We also solve explicitly the hierarchy for the four-point pion amplitudes (one loop). In section 5 the technique

for the determination of the higher order solution is developed. In section 6 we apply this technique on some examples at the two loop level. In particular we obtain the solution of the hierarchy for the four point pion functions at two loops. In section 7 we comment on the possible finite renormalizations which are allowed from a mathematical point of view by the weak power-counting, order by order in the loop expansion, and we show that they can be interpreted as a redefinition of the external sources  $J_{a\mu}$  and  $K_0$  by finite quantum corrections. Conclusions are finally given in section 8.

## 2. The flat connection formalism

In the flat connection formalism [1] the pion fields are embedded into the SU(2) flat connection

$$F_\mu = i\Omega\partial_\mu\Omega^\dagger = \frac{1}{2}F_{a\mu}\tau_a. \tag{2.1}$$

In the above equation  $\tau_a$  are the Pauli matrices and  $\Omega$  denotes the SU(2) group element.  $\Omega$  is parameterized in terms of the pion fields  $\phi_a$  as follows:

$$\begin{aligned} \Omega = \frac{1}{v_D}(\phi_0 + i\tau_a\phi_a), \quad \Omega^\dagger\Omega = 1, \quad \det \Omega = 1, \\ \phi_0^2 + \phi_a^2 = v_D^2. \end{aligned} \tag{2.2}$$

$v_D$  is the  $D$ -dimensional mass scale

$$v_D = v^{D/2-1}$$

and  $v$  has mass dimension one.

The  $D$ -dimensional action of the nonlinear sigma model is written in the presence of an external vector source  $J_{a\mu}$ <sup>1</sup> and of a scalar source  $K_0$  coupled to the solution of the nonlinear sigma model constraint  $\phi_0$ :

$$\Gamma^{(0)} = \int d^Dx \left( \frac{v_D^2}{8}(F_{a\mu} - J_{a\mu})^2 + K_0\phi_0 \right). \tag{2.3}$$

The invariance of the Haar measure in the path-integral under the local gauge transformations

$$\begin{aligned} \Omega' &= U\Omega, \\ F'_\mu &= UF_\mu U^\dagger + iU\partial_\mu U^\dagger, \end{aligned} \tag{2.4}$$

where  $U$  is an element of SU(2) allows to derive the following local functional equation for the 1-PI vertex functional  $\Gamma$  [1]

$$\left( -\partial_\mu \frac{\delta\Gamma}{\delta J_{a\mu}} + \epsilon_{abc} J_{c\mu} \frac{\delta\Gamma}{\delta J_{b\mu}} + \frac{1}{2} K_0 \phi_a + \frac{1}{2} \frac{\delta\Gamma}{\delta K_0} \frac{\delta\Gamma}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta\Gamma}{\delta \phi_b} \right)(x) = 0. \tag{2.5}$$

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<sup>1</sup>In this paper we denote by  $J_{a\mu}$  the background connection. The classical action of ref. [1] differs by a term  $\frac{v_D^2}{8} J_{a\mu}^2$  w.r.t. the action in eq. (2.3). The source coupled to the flat connection is given by  $-\frac{v_D^2}{4} J_{a\mu}$ . Moreover we set the gauge coupling constant  $g$  to 1.

Moreover one requires that the vacuum expectation value of the order parameter is fixed by the condition

$$\left. \frac{\delta\Gamma}{\delta K_0(x)} \right|_{\vec{\phi}=K_0=J_{a\mu}=0} = v_D. \quad (2.6)$$

A weak power-counting theorem [2] exists for the loop-wise perturbative expansion. Accordingly at any given loop order the number of divergent ancestor amplitudes (i.e. those only involving the insertion of the ancestor composite operators) is finite. On the contrary, already at one loop level there is an infinite number of divergent 1-PI amplitudes involving the  $\phi_a$  fields (descendant amplitudes). The latter can be fixed in terms of the ancestor ones by recursively differentiating the local functional equation (2.5).

### 3. One-loop solution

In the one-loop approximation eq. (2.5) becomes

$$\mathcal{S}_a(\Gamma^{(1)}) = \left( -\partial_\mu \frac{\delta\Gamma^{(1)}}{\delta J_{a\mu}} + \epsilon_{abc} J_{c\mu} \frac{\delta\Gamma^{(1)}}{\delta J_{b\mu}} + \frac{1}{2} \frac{\delta\Gamma^{(0)}}{\delta K_0} \frac{\delta\Gamma^{(1)}}{\delta\phi_a} + \frac{1}{2} \frac{\delta\Gamma^{(1)}}{\delta K_0} \frac{\delta\Gamma^{(0)}}{\delta\phi_a} + \frac{1}{2} \epsilon_{abc} \phi_c \frac{\delta\Gamma^{(1)}}{\delta\phi_b} \right)(x) = 0. \quad (3.1)$$

In order to solve the above equation we construct invariant variables in one-to-one correspondence with the external sources. For that purpose we remark that the combination

$$\bar{K}_0 = \frac{v_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta\phi_a} \quad (3.2)$$

with

$$S_0 = \frac{v_D^2}{8} \int d^D x \left( F_{a\mu} - J_{a\mu} \right)^2 \quad (3.3)$$

is an invariant [2]. Moreover the transformation  $K_0 \rightarrow \bar{K}_0$  is invertible.

On the other hand eq. (3.1) implies that  $J_{a\mu}$  transforms as a background connection.

The transformation properties of  $\phi_a$  implement the non-linearly realized SU(2) local transformation in eq. (2.4). Hence  $F_{a\mu}$  transforms as a gauge connection and therefore the combination

$$I_\mu = I_{a\mu} \frac{\tau_a}{2} = F_\mu - J_\mu \quad (3.4)$$

transforms in the adjoint representation (being the difference of two connections):

$$I'_\mu = U I_\mu U^\dagger. \quad (3.5)$$

As a consequence the conjugate of  $I_\mu$  w.r.t.  $\Omega$

$$j_\mu = j_{a\mu} \frac{\tau_a}{2} = \Omega^\dagger I_\mu \Omega \quad (3.6)$$

is an invariant under the transformations in eqs. (2.4) and (3.5).

By direct computation one finds that  $j_{a\mu}$  in eq. (3.6) is given by

$$\begin{aligned} v_D^2 j_{a\mu} &= v_D^2 I_{a\mu} - 2\phi_b^2 I_{a\mu} + 2\phi_b I_{b\mu} \phi_a + 2\phi_0 \epsilon_{abc} \phi_b I_{c\mu} \\ &\equiv v_D^2 R_{ba} I_{b\mu}. \end{aligned} \tag{3.7}$$

The matrix  $R_{ba}$  in the above equation is an element of the adjoint representation of the SU(2) group. Hence the transformation  $J_{a\mu} \rightarrow j_{a\mu}$  is invertible.

The linearized functional equation (3.1) has a very simple form in the variables  $\{\phi_a, \bar{K}_0, j_{a\mu}\}$ . In fact, by taking into account the invariance of  $\bar{K}_0$  and  $j_{a\mu}$  under  $\mathcal{S}_a$ , eq. (3.1) reduces to

$$\Theta_{ab} \frac{\delta\Gamma^{(1)}[\phi_a, \bar{K}_0, j_{a\mu}]}{\delta\phi_b} = 0, \tag{3.8}$$

where the matrix  $\Theta_{ab}$  gives the variation of  $\phi_b$ :

$$\Theta_{ab} = \frac{1}{2}\phi_0 \delta_{ab} + \frac{1}{2}\epsilon_{abc} \phi_c. \tag{3.9}$$

$\Theta_{ab}$  is invertible as a consequence of the nonlinear constraint in the second line of eq. (2.2) and thus eq. (3.8) is equivalent to

$$\frac{\delta\Gamma^{(1)}[\phi_a, \bar{K}_0, j_{a\mu}]}{\delta\phi_b} = 0. \tag{3.10}$$

That means that the only dependence of the symmetric vertex functional  $\Gamma^{(1)}$  on the pion fields is through the variables  $\bar{K}_0$  and  $j_{a\mu}$ .

This in turn allows to integrate the linearized functional equation (3.1). For that purpose one has to replace in the ancestor amplitudes 1-PI functional the source  $K_0$  with  $\frac{1}{v_D}\bar{K}_0$  and  $J_{a\mu}$  with  $-j_{a\mu}$ . The normalization of  $\bar{K}_0$  and  $j_{a\mu}$  is fixed by the boundary conditions

$$\begin{aligned} \bar{K}_0|_{\vec{\phi}=0} &= v_D K_0 \\ -j_{a\mu}|_{\vec{\phi}=0} &= J_{a\mu}. \end{aligned} \tag{3.11}$$

By eq. (3.10) this algorithm gives rise to the full dependence on the pion fields at the one loop level. Thus we can state the following Proposition:

**Proposition 1.** Given the ancestor amplitudes 1-PI functional  $\mathcal{A}^{(1)}[K_0, J_{a\mu}]$  the solution of the linearized local functional equation (3.1) is obtained through the replacement rule

$$\Gamma^{(1)}[\phi_a, K_0, J_{a\mu}] = \mathcal{A}^{(1)}[K_0, J_{a\mu}] \Big|_{K_0 \rightarrow \frac{1}{v_D}\bar{K}_0, J_{a\mu} \rightarrow -j_{a\mu}} \tag{3.12}$$

where in the r.h.s. of the above equation  $\bar{K}_0$  is given by eq. (3.2) and  $j_{a\mu}$  by eq. (3.7).

In view of this result we say that  $\Gamma^{(1)}$  depends on the  $\phi$ 's only implicitly (i.e. through  $\bar{K}_0$  and  $j_{a\mu}$ ). This terminology will prove convenient when studying the dependence of the vertex functional on the  $\phi$ 's at higher orders.

We stress that no restriction to the space of local functionals is used in the above derivation. Eq. (3.12) thus provides the full set of Green functions involving at least one pion in terms of the ancestor amplitudes. This solves the hierarchy at the one loop level.

#### 4. One-loop examples

When restricted to the local (in the sense of formal power series) functionals, the prescription in eq. (3.12) gives back the results of [2]. This follows from the uniqueness of the hierarchy solution once the ancestor amplitudes are fixed.

As an example we derive the local invariants  $\mathcal{I}_1, \dots, \mathcal{I}_7$  parameterizing the one-loop divergences of the nonlinear sigma model in  $D = 4$  (see appendix C) by performing the substitution  $K_0 \rightarrow \frac{1}{v_D} \overline{K}_0$ ,  $J_{a\mu} \rightarrow -j_{a\mu}$  in the relevant ancestor monomials

$$\begin{aligned} & \int \partial_\mu J_{a\nu} \partial^\mu J_a^\nu, \quad \int \partial J_a \partial J_a, \quad \int \epsilon_{abc} \partial_\mu J_{a\nu} J_b^\mu J_c^\nu, \\ & \int K_0^2, \quad \int K_0 J^2, \quad \int (J^2)^2, \quad \int J_{a\mu} J_b^\mu J_{a\nu} J_b^\nu. \end{aligned} \quad (4.1)$$

The monomials in the second line of the above equation do not contain derivatives. By using the SU(2) constraint

$$R_{ba} R_{ca} = \delta_{bc} \quad (4.2)$$

we get

$$j_{a\mu}^2 = I_{a\mu}^2, \quad j_{a\mu} j_b^\mu j_{a\nu} j_b^\nu = I_{a\mu} I_b^\mu I_{a\nu} I_b^\nu. \quad (4.3)$$

Therefore

$$\begin{aligned} \int d^D x K_0^2 & \rightarrow \frac{1}{v_D^2} \int d^D x \overline{K}_0^2 = \frac{1}{v_D^2} \mathcal{I}_4, \\ \int d^D x K_0 J^2 & \rightarrow \frac{1}{v_D} \int d^D x \overline{K}_0 j^2 = \frac{1}{v_D} \mathcal{I}_5, \\ \int d^D x (J^2)^2 & \rightarrow \int d^D x (j^2)^2 = \mathcal{I}_6, \\ \int d^D x J_{a\mu} J_b^\mu J_{b\nu} J_b^\nu & \rightarrow \int d^D x j_{a\mu} j_b^\mu j_{b\nu} j_b^\nu = \mathcal{I}_7. \end{aligned} \quad (4.4)$$

In order to establish the matching for the ancestor monomials involving derivatives in the first line of eq. (4.1), we notice that the flat connection  $F_{a\mu}$  can be computed in terms of  $R_{ba}$  as well (since  $R_{ba}$  belongs to the adjoint representation of the SU(2) group). In fact one finds

$$i R_{bc} \partial_\mu R_{ca}^\dagger = i R_{bc} \partial_\mu R_{ac} = (T_c)_{ba} F_{c\mu} \quad (4.5)$$

where  $(T_c)_{ba} = i \epsilon_{cab}$  are the generators of the adjoint representation satisfying the commutation relation

$$[T_a, T_b] = i \epsilon_{abc} T_c. \quad (4.6)$$

Eq. (4.5) can be checked as follows. We set

$$\begin{aligned} R_a & \equiv \Omega^\dagger \tau_a \Omega = \tau_b R_{ab} \\ R_{ab} & = \frac{1}{2} \text{Tr} (\tau_b \Omega^\dagger \tau_a \Omega). \end{aligned} \quad (4.7)$$

By using the following identities

$$\begin{aligned}
T_r(\tau_a F_\mu \tau_b) &= T_r(\Omega^\dagger \tau_a F_\mu \tau_b \Omega) \\
&= iT_r(\Omega^\dagger \tau_a \Omega \partial_\mu \Omega^\dagger \tau_b \Omega) \\
&= iT_r(\Omega^\dagger \tau_a \Omega \partial_\mu [\Omega^\dagger \tau_b \Omega]) - iT_r(\Omega^\dagger \tau_a \Omega \Omega^\dagger \tau_b \partial_\mu \Omega) \\
&= iT_r(\Omega^\dagger \tau_a \Omega \partial_\mu [\Omega^\dagger \tau_b \Omega]) + iT_r(\tau_a \Omega \Omega^\dagger \tau_b \Omega \partial_\mu \Omega^\dagger)
\end{aligned} \tag{4.8}$$

we find

$$T_r(\tau_a F_\mu \tau_b) - T_r(\tau_b F_\mu \tau_a) = iT_r(\Omega^\dagger \tau_a \Omega \partial_\mu [\Omega^\dagger \tau_b \Omega]) \tag{4.9}$$

which gives directly eq. (4.5):

$$-i\epsilon_{abc} F_{c\mu} = iR_{ac} \partial_\mu R_{bc} = -iR_{bc} \partial_\mu R_{ac}. \tag{4.10}$$

By repeated application of eq. (4.2) and eq. (4.5) we then get

$$\begin{aligned}
\int d^D x \partial_\mu J_{a\nu} \partial^\mu J_a^\nu &\rightarrow \int d^D x \partial_\mu j_{a\nu} \partial^\mu j_a^\nu = \int d^D x \partial_\mu (R_{ba} I_{\nu b}) \partial^\mu (R_{ca} I_c^\nu) \\
&= \int d^D x (D_\mu [F] I_\nu)_a (D^\mu [F] I^\nu)_a = \mathcal{I}_1,
\end{aligned} \tag{4.11}$$

where  $D_\mu [F]$  is the covariant derivative w.r.t.  $F_{a\mu}$ :

$$(D_\mu [F] I_\nu)_a = \partial_\mu I_{a\nu} + \epsilon_{abc} F_{b\mu} I_{c\nu}. \tag{4.12}$$

In a similar way we get

$$\int d^D x \partial J_a \partial J_a \rightarrow \int d^D x \partial j_a \partial j_a = \int d^D x (D_\mu [F] I^\mu)_a (D_\nu [F] I^\nu)_a = \mathcal{I}_2. \tag{4.13}$$

Moreover

$$\begin{aligned}
\int d^D x \epsilon_{abc} \partial_\mu J_{a\nu} J_b^\mu J_c^\nu &\rightarrow - \int d^D x \epsilon_{abc} \partial_\mu j_{a\nu} j_b^\mu j_c^\nu \\
&= - \int d^D x \epsilon_{abc} \left( \partial_\mu R_{qa} I_{q\nu} R_{pb} I_p^\mu R_{rc} I_r^\nu + R_{qa} \partial_\mu I_{q\nu} R_{pb} I_p^\mu R_{rc} I_r^\nu \right).
\end{aligned} \tag{4.14}$$

By noticing that

$$\epsilon_{abc} R_{qa} R_{pb} R_{rc} = \epsilon_{qpr} \tag{4.15}$$

and by using eqs. (4.2) and (4.5) into eq. (4.14) we finally get

$$- \int d^D x \epsilon_{abc} \partial_\mu j_{a\nu} j_b^\mu j_c^\nu = - \int d^D x \epsilon_{abc} (D_\mu [F] I_\nu)_a I_b^\mu I_c^\nu = -\mathcal{I}_3. \tag{4.16}$$

As we have mentioned several times, the algorithm for solving the hierarchy based on Proposition 1 can be applied in order to derive the full Green functions involving at



least one pion field in terms of the ancestor amplitudes. As an example, we obtain here the full one-loop four point pion amplitude in terms of the relevant ancestor amplitudes  $\Gamma_{JJ}^{(1)}, \Gamma_{JJJ}^{(1)}, \Gamma_{JJJJ}^{(1)}, \Gamma_{K_0K_0}^{(1)}$  and  $\Gamma_{K_0JJ}^{(1)}$ . For that purpose one has to perform the substitution  $J_{a\mu} \rightarrow -j_{a\mu}$  and  $K_0 \rightarrow \frac{1}{v_D} \overline{K}_0$  in the relevant part of the ancestor functional

$$\begin{aligned}
 \mathcal{A}^{(1)}[K_0, J_{a\mu}] &= \frac{1}{2} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)}^{(1)} J_{a\mu}(x) J_{b\nu}(y) + \\
 &+ \frac{1}{3!} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)J_{c\rho}(z)}^{(1)} J_{a\mu}(x) J_{b\nu}(y) J_{c\rho}(z) \\
 &+ \frac{1}{4!} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)J_{c\rho}(z)J_{d\sigma}(w)}^{(1)} J_{a\mu}(x) J_{b\nu}(y) J_{c\rho}(z) J_{d\sigma}(w) \\
 &+ \frac{1}{2} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)K_0(z)}^{(1)} J_{a\mu}(x) J_{b\nu}(y) K_0(z) \\
 &+ \frac{1}{2} \int \Gamma_{K_0(x)K_0(y)}^{(1)} K_0(x) K_0(y) + \dots
 \end{aligned} \tag{4.17}$$

by keeping all terms contributing up to four pion fields. This amounts to truncate the expansion of  $\overline{K}_0$  up to two  $\phi$ 's and the expansion of  $j_{a\mu}$  up to three  $\phi$ 's:

$$\begin{aligned}
 \overline{K}_0 &= \frac{1}{v_D} \phi_a \square \phi_a + \dots, \\
 j_{a\mu} &= \frac{2}{v_D} \partial_\mu \phi_a - \frac{2}{v_D^2} \epsilon_{abc} \partial_\mu \phi_b \phi_c + \frac{1}{v_D^3} \left( -\phi_b^2 \partial_\mu \phi_a + 2\phi_b \partial_\mu \phi_b \phi_a \right) + \dots
 \end{aligned} \tag{4.18}$$

Then one gets

$$\begin{aligned}
 \Gamma^{(1)}[\phi\phi\phi\phi] &= \frac{2}{v_D^4} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)}^{(1)} \left( \partial_\mu \phi_a(x) (-\phi_c^2(y) \partial_\nu \phi_b(y) + 2\phi_c(y) \partial_\nu \phi_c(y) \phi_b(y)) \right. \\
 &\quad \left. + \epsilon_{apq} \epsilon_{brs} \partial_\mu \phi_p(x) \phi_q(x) \partial_\nu \phi_r(y) \phi_s(y) \right) \\
 &+ \frac{4}{v_D^4} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)J_{c\rho}(z)}^{(1)} \epsilon_{apq} \partial_\mu \phi_p(x) \phi_q(x) \partial_\nu \phi_b(y) \partial_\rho \phi_c(z) \\
 &+ \frac{2}{3v_D^4} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)J_{c\rho}(z)J_{d\sigma}(w)}^{(1)} \partial_\mu \phi_a(x) \partial_\nu \phi_b(y) \partial_\rho \phi_c(z) \partial_\sigma \phi_d(w) \\
 &+ \frac{2}{v_D^3} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)K_0(z)}^{(1)} \partial_\mu \phi_a(x) \partial_\nu \phi_b(y) (\phi_c \square \phi_c)(z) \\
 &+ \frac{1}{2v_D^2} \int \Gamma_{K_0(x)K_0(y)}^{(1)} (\phi_a \square \phi_a)(x) (\phi_b \square \phi_b)(y).
 \end{aligned} \tag{4.19}$$

## 5. Higher orders

At higher orders the functional equation (2.5) yields an inhomogeneous equation for  $\Gamma^{(n)}$ ,  $n > 1$ :

$$\mathcal{S}_a(\Gamma^{(n)}) = -\frac{1}{2} \sum_{j=1}^{n-1} \frac{\delta\Gamma^{(j)}}{\delta K_0} \frac{\delta\Gamma^{(n-j)}}{\delta\phi_a}. \tag{5.1}$$

In order to recursively integrate eq. (5.1) order by order in the loop expansion it is convenient to introduce a BRST formulation for the linearized functional operator  $\mathcal{S}_a$ . For that purpose we define the BRST differential [2]

$$s = \int d^D x \omega_a \mathcal{S}_a$$

where  $\omega_a$  are classical anticommuting local parameters. The variables  $\bar{K}_0$  and  $j_{a\mu}$  are  $s$ -invariant while

$$s\phi_a = \frac{1}{2}\phi_0\omega_a + \frac{1}{2}\epsilon_{abc}\phi_b\omega_c \equiv \Theta_{ab}\omega_b, \quad s\omega_a = -\frac{1}{2}\epsilon_{abc}\omega_b\omega_c. \quad (5.2)$$

The BRST transformation of  $\omega_a$  is dictated by nilpotency.  $\omega_a$  have ghost number one, while all the remaining variables have ghost number zero. In view of the fact that there are no variables with negative ghost number and that the vertex functional  $\Gamma$  has ghost number zero,  $\Gamma$  cannot depend on  $\omega_a$ .

The introduction of a BRST differential allows to make use of the technique of the Slavnov-Taylor (ST) parameterization of the effective action [7]–[9] (originally developed in order to restore the ST identities for power-counting renormalizable gauge theories in the absence of a symmetric regularization) in order to solve the local functional equation at orders  $\geq 1$ .

For that purpose we remark that, since the matrix  $\Theta_{ab}$  in eq. (5.2) is invertible, we can perform a further change of variables by setting

$$\bar{\omega}_a = \Theta_{ab}\omega_b. \quad (5.3)$$

The inverse matrix  $\Theta_{ca}^{-1}$  is given by

$$\Theta_{ca}^{-1} = \frac{2\phi_0}{v_D^2}\delta_{ca} + \frac{2}{v_D^2\phi_0}\phi_c\phi_a - \frac{2}{v_D^2}\epsilon_{cpa}\phi_p. \quad (5.4)$$

The action of  $s$  on the variables  $\{\bar{K}_0, j_{a\mu}, \phi_a, \bar{\omega}_a\}$  is finally given by

$$\begin{aligned} s\bar{K}_0 &= sj_{a\mu} = 0, \\ s\phi_a &= \bar{\omega}_a, \quad s\bar{\omega}_a = 0, \end{aligned} \quad (5.5)$$

i.e.  $s$  has been cohomologically trivialized:  $(\phi_a, \bar{\omega}_a)$  form a BRST doublet [14]–[16], while  $\bar{K}_0$  and  $j_{a\mu}$  are invariant.

We are now in a position to recursively solve the local functional equation at higher orders in perturbation theory. By using the BRST differential  $s$  eq. (5.1) reads

$$s\Gamma^{(n)} = \Delta^{(n)} \equiv -\frac{1}{2}\sum_{j=1}^{n-1}\int d^D x \omega_a \frac{\delta\Gamma^{(j)}}{\delta K_0(x)} \frac{\delta\Gamma^{(n-j)}}{\delta\phi_a(x)}, \quad (5.6)$$

where  $\Delta^{(n)}$  depends only on known lower order terms. Nilpotency of  $s$  implies that  $\Delta^{(n)}$  is invariant:

$$s\Delta^{(n)} = 0. \quad (5.7)$$

This consistency condition can be checked to hold as a consequence of the fulfillment of the functional equation up to order  $n - 1$ , as shown in appendix A.

By using eq. (5.5) into eq. (5.6) we find

$$\int d^D x \bar{\omega}_a \frac{\delta \Gamma^{(n)}}{\delta \phi_a} = \Delta^{(n)}[\bar{\omega}_a, \phi_a, \bar{K}_0, j_{a\mu}]. \quad (5.8)$$

We remark that  $\Delta^{(n)}$  is linear in  $\bar{\omega}_a$ . By differentiating eq. (5.8) and by setting  $\bar{\omega}_a = 0$  we get

$$\frac{\delta \Gamma^{(n)}}{\delta \phi_a(x)} = \frac{\delta \Delta^{(n)}}{\delta \bar{\omega}_a(x)} \quad (5.9)$$

which fixes the explicit dependence of the symmetric vertex functional  $\Gamma^{(n)}$  on  $\phi_a(x)$  ( $\Gamma^{(n)}$  depends on  $\phi$  also implicitly through the invariant variables  $j_{a\mu}$  and  $\bar{K}_0$ ). By successive differentiation of eq. (5.9) we obtain

$$\begin{aligned} \Gamma_{\phi_{a_1} \dots \phi_{a_m} \zeta_{b_1} \dots \zeta_{b_n}}^{(n)} &= \Delta_{\bar{\omega}_{a_1} \phi_{a_2} \dots \phi_{a_m} \zeta_{b_1} \dots \zeta_{b_n}}^{(n)} \\ &= \frac{1}{m!} \sum_{\sigma \in S_m} \Delta_{\bar{\omega}_{a_{\sigma(1)}} \phi_{a_{\sigma(2)}} \dots \phi_{a_{\sigma(m)}} \zeta_{b_1} \dots \zeta_{b_n}}^{(n)} \end{aligned} \quad (5.10)$$

where  $\zeta$  is a collective notation standing for  $j_{a\mu}$  and  $\bar{K}_0$ .

The equality in the second line of the above equation is a consequence of the Bose statistics of the  $\phi$ 's. We point out that eq. (5.10) imposes a consistency condition on  $\Delta^{(n)}$ , i.e.

$$\Delta_{\bar{\omega}_{a_1} \phi_{a_2} \dots \phi_{a_m} \zeta_{b_1} \dots \zeta_{b_n}}^{(n)} = \frac{1}{m!} \sum_{\sigma \in S_m} \Delta_{\bar{\omega}_{a_{\sigma(1)}} \phi_{a_{\sigma(2)}} \dots \phi_{a_{\sigma(m)}} \zeta_{b_1} \dots \zeta_{b_n}}^{(n)}. \quad (5.11)$$

This condition holds as a consequence of eq. (5.7), as is proven in appendix B.

Eq. (5.9) shows that at order  $n \geq 2$  the vertex functional exhibits a further dependence on the  $\phi$ 's (in addition to the implicit one through the variables  $\bar{K}_0$  and  $j_{a\mu}$ ). We refer to it as the explicit dependence of  $\Gamma^{(n)}$  on  $\phi_a$ . It is a remarkable fact that this latter dependence on the pion fields comes from amplitudes involving the pion field of lower order strictly. In particular, they do not affect the  $n$ -th loop ancestor amplitudes.

In order to recover the full  $n$ -th loop vertex functional one also needs to take into account the implicit dependence on the pion fields through  $\bar{K}_0$  and  $j_{a\mu}$ . In fact we can state the following

**Proposition 2.** Given the functional  $\mathcal{A}^{(n)}[K_0, J_{a\mu}]$  collecting the  $n$ -th order ancestor amplitudes,  $n \geq 2$ , the full  $n$ -th loop vertex functional is given by

$$\begin{aligned} \Gamma^{(n)}[\phi_a, K_0, J_{a\mu}] &= \mathcal{A}^{(n)}[K_0, J_{a\mu}] \Big|_{K_0 \rightarrow \frac{1}{v_D} \bar{K}_0, J_{a\mu} \rightarrow -j_{a\mu}} \\ &+ \int d^D x \int_0^1 dt \phi_a(x) \lambda_t \frac{\delta \Delta^{(n)}}{\delta \bar{\omega}_a(x)} \end{aligned} \quad (5.12)$$

where  $\lambda_t$  acts as follows on a functional  $X[\phi_a, \overline{K}_0, j_{a\mu}]$ :

$$\lambda_t X[\phi_a, \overline{K}_0, j_{a\mu}] = X[t\phi_a, \overline{K}_0, j_{a\mu}]. \quad (5.13)$$

The first term in the r.h.s. of eq. (5.12) accounts for the implicit dependence on  $\phi_a$  through  $\overline{K}_0$  and  $j_{a\mu}$ . It is of the same form as in the one loop approximation eq. (3.12).

The second term in the r.h.s. of eq. (5.12) is present only from two loops on. It arises as a consequence of the bilinearity of the local functional equation (2.5). It gives rise to the explicit dependence of  $\Gamma^{(n)}$  on  $\phi_a$  dictated by eq. (5.9). This can be checked by taking derivatives w.r.t.  $\phi_{a_1}, \dots, \phi_{a_m}$  of eq. (5.12) and then setting  $\phi = 0$  (derivatives w.r.t.  $\zeta_{b_1}, \dots, \zeta_{b_n}$  do not play any role in the following argument). The only contribution comes from the second term and yields

$$\begin{aligned} \frac{\delta^m}{\delta\phi_{a_1} \dots \delta\phi_{a_m}} \int d^D x \int_0^1 dt \phi_a(x) \lambda_t \frac{\delta\Delta^{(n)}}{\delta\overline{\omega}_a(x)} &= \frac{\delta^m}{\delta\phi_{a_1} \dots \delta\phi_{a_m}} \int d^D x d^D y_1 \dots d^D y_{m-1} \\ &\frac{1}{(m-1)!} \int_0^1 dt t^{m-1} \Delta_{\overline{\omega}_a(x)\phi_{b_1}(y_1)\dots\phi_{b_{m-1}}(y_{m-1})}^{(n)} \phi_a(x) \phi_{b_1}(y_1) \dots \phi_{b_{m-1}}(y_{m-1}) \\ &= \frac{1}{m!} \sum_{\sigma \in S_m} \Delta_{\overline{\omega}_{a_{\sigma(1)}}\phi_{a_{\sigma(2)}}\dots\phi_{a_{\sigma(m)}}}^{(n)} = \Gamma_{\phi_{a_1}\dots\phi_{a_m}}^{(n)} \end{aligned} \quad (5.14)$$

where in the last line we have used eq. (5.10).

Eq. (5.12) provides the full set of  $n$ -th order Green functions in terms of  $n$ -th order ancestor amplitudes and known lower order terms, thus solving the hierarchy.

## 6. Two-loop examples

In this section we apply the method developed in section 5 at two loop order. The two-loop inhomogeneous term is

$$\Delta^{(2)} = - \int d^D x \frac{1}{2} \omega_a(x) \frac{\delta\Gamma^{(1)}}{\delta K_0(x)} \frac{\delta\Gamma^{(1)}}{\delta\phi_a(x)}. \quad (6.1)$$

In order to apply eq. (5.12) we need to express the r.h.s. in terms of the variables  $\{\overline{K}_0, j_{a\mu}, \phi_a\}$ . For that purpose we write

$$\begin{aligned} \Delta^{(2)} &= - \int d^D x \frac{1}{2} \Theta_{ab}^{-1} \overline{\omega}_b \int d^D y \frac{\delta\overline{K}_0(y)}{\delta K_0(x)} \frac{\delta\Gamma^{(1)}}{\delta\overline{K}_0(y)} \\ &\int d^D z \left( \frac{\delta\overline{K}_0(z)}{\delta\phi_a(x)} \frac{\delta}{\delta\overline{K}_0(z)} + \frac{\delta j_{c\mu}(z)}{\delta\phi_a(x)} \frac{\delta}{\delta j_{c\mu}(z)} + \delta^D(x-z) \frac{\delta}{\delta\phi_a(z)} \right) \Gamma^{(1)}. \end{aligned} \quad (6.2)$$

where the matrix  $\Theta_{ab}^{-1}$  is given in eq. (5.4).

Moreover

$$\frac{\delta\overline{K}_0(y)}{\delta K_0(x)} = \frac{v_D^2}{\phi_0} \delta^D(y-x) \quad (6.3)$$

while by eq. (3.10) one has in the variables  $\{\bar{K}_0, j_{a\mu}, \phi_a\}$

$$\frac{\delta\Gamma^{(1)}}{\delta\phi_a(x)} = 0. \quad (6.4)$$

Therefore

$$\begin{aligned} \Delta^{(2)} = & - \int d^D x \frac{1}{2} \frac{v_D^2}{\phi_0} \Theta_{ab}^{-1} \bar{\omega}_b \frac{\delta\Gamma^{(1)}}{\delta\bar{K}_0(x)} \\ & \int d^D z \left( \frac{\delta\bar{K}_0(z)}{\delta\phi_a(x)} \frac{\delta}{\delta\bar{K}_0(z)} + \frac{\delta j_{c\mu}(z)}{\delta\phi_a(x)} \frac{\delta}{\delta j_{c\mu}(z)} \right) \Gamma^{(1)} \end{aligned} \quad (6.5)$$

It is useful to introduce two transition functions (encoding the effect of the change of variables from  $\{K_0, J_{a\mu}, \phi_a\}$  to  $\{\bar{K}_0, j_{a\mu}, \phi_a\}$ ):

$$\begin{aligned} G_b(x, z) &= \frac{1}{2} \frac{v_D^2}{\phi_0(x)} \Theta_{ab}^{-1}(x) \frac{\delta\bar{K}_0(z)}{\delta\phi_a(x)}, \\ H_{bc,\mu}(x, z) &= \frac{1}{2} \frac{v_D^2}{\phi_0(x)} \Theta_{ab}^{-1}(x) \frac{\delta j_{c\mu}(z)}{\delta\phi_a(x)} \end{aligned} \quad (6.6)$$

so that eq. (6.3) reads

$$\Delta^{(2)} = - \int d^D x \int d^D z \bar{\omega}_b(x) \frac{\delta\Gamma^{(1)}}{\delta\bar{K}_0(x)} \left( G_b(x, z) \frac{\delta}{\delta\bar{K}_0(z)} + H_{bc,\mu}(x, z) \frac{\delta}{\delta j_{c\mu}(z)} \right) \Gamma^{(1)}. \quad (6.7)$$

In the two-loop approximation eq. (5.9) is finally

$$\begin{aligned} \frac{\delta\Gamma^{(2)}}{\delta\phi_b(x)} &= \frac{\delta\Delta^{(2)}}{\delta\bar{\omega}_b(x)} \\ &= - \int d^D z \frac{\delta\Gamma^{(1)}}{\delta\bar{K}_0(x)} \left( G_b(x, z) \frac{\delta}{\delta\bar{K}_0(z)} + H_{bc,\mu}(x, z) \frac{\delta}{\delta j_{c\mu}(z)} \right) \Gamma^{(1)} \end{aligned} \quad (6.8)$$

while eq. (5.12) consequently reads

$$\begin{aligned} \Gamma^{(2)}[\phi_a, K_0, J_{a\mu}] &= \mathcal{A}^{(2)}[K_0, J_{a\mu}] \Big|_{K_0 \rightarrow \frac{1}{v_D} \bar{K}_0, J_{a\mu} \rightarrow -j_{a\mu}} \\ & - \int d^D x \int_0^1 dt \phi_b(x) \lambda_t \int d^D z \frac{\delta\Gamma^{(1)}}{\delta\bar{K}_0(x)} \left( G_b(x, z) \frac{\delta}{\delta\bar{K}_0(z)} + H_{bc,\mu}(x, z) \frac{\delta}{\delta j_{c\mu}(z)} \right) \Gamma^{(1)}. \end{aligned} \quad (6.9)$$

The second line encodes the effects of the nonlinearity of the local functional equation at two loop order.

It should be noticed that, due to the peculiar structure of the dependence of the one-loop vertex functional on the pions given by eq. (3.10), one finds some special simplifications at two loop level. In particular the second line of eq. (6.9) does not contribute to the four point pion Green function in the two loop approximation.

In order to show this property we remark that the expansion of  $\bar{K}_0$  starts with two  $\phi$ 's while  $j_{a\mu}$  starts with one  $\phi$ . Hence the term with two derivatives w.r.t.  $\bar{K}_0$  in the second line of eq. (6.9) gives contributions of order  $O(\phi^5)$ .

In order to obtain the contribution to the four point pion function of the term involving one derivative w.r.t.  $j_{c\mu}$  in the second line it is sufficient to keep  $H_{bc,\mu}$  at order zero:

$$H_{bc;\mu}(x, z) = \frac{2}{v_D} \delta_{bc} \partial_{z\mu} \delta^D(x - z) + O(\phi). \quad (6.10)$$

This yields

$$-\frac{2}{v_D} \int d^D x \phi_b(x) \left[ \frac{\delta\Gamma^{(1)}}{\delta\bar{K}_0(x)} \right]_{\phi\phi} \partial^\mu \left[ \frac{\delta\Gamma^{(1)}}{\delta j_{b\mu}(x)} \right]_{\phi} \quad (6.11)$$

where the subscript denotes the order of the projection for the  $\phi$ 's. Moreover the derivative

$$\left[ \frac{\delta\Gamma^{(1)}}{\delta j_{b\mu}(x)} \right]_{\phi} \quad (6.12)$$

receives contributions only from the amplitude  $\Gamma_{J_{a\mu}J_{b\nu}}^{(1)}$  [1] through

$$\begin{aligned} \frac{1}{2} \int d^D x d^D y \Gamma_{J_{a\mu}(x)J_{b\nu}(y)}^{(1)} j_{a\mu}(x) j_{b\nu}(y) &= -\frac{1}{2} \int d^D x d^D y (\square g^{\mu\nu} - \partial^\mu \partial^\nu) j_{a\mu}(x) j_{b\nu}(y) \\ &\int d^D p \frac{4i}{m_D^4} \frac{1}{D-1} e^{ip(x-y)} I_2(p) \end{aligned} \quad (6.13)$$

where

$$I_2(p) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k+p)^2}. \quad (6.14)$$

By taking the gradient according to eq. (6.11) one finds zero as a consequence of the transversality of  $\Gamma_{J_{a\mu}(x)J_{b\nu}(y)}^{(1)}$ . Therefore the second line of eq. (6.9) does not give any contribution to the four point pion function at two loop level. The contribution from the first line can be derived according to the methods discussed in section 4. So we get finally

$$\begin{aligned} \Gamma^{(2)}[\phi\phi\phi\phi] &= \frac{2}{v_D^4} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)}^{(2)} \left( \partial_\mu \phi_a(x) (-\phi_c^2(y) \partial_\nu \phi_b(y) + 2\phi_c(y) \partial_\nu \phi_c(y) \phi_b(y)) \right. \\ &\quad \left. + \epsilon_{apq} \epsilon_{brs} \partial_\mu \phi_p(x) \phi_q(x) \partial_\nu \phi_r(y) \phi_s(y) \right) \\ &+ \frac{4}{v_D^4} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)J_{c\rho}(z)}^{(2)} \epsilon_{apq} \partial_\mu \phi_p(x) \phi_q(x) \partial_\nu \phi_b(y) \partial_\rho \phi_c(z) \\ &+ \frac{2}{3v_D^4} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)J_{c\rho}(z)J_{d\sigma}(w)}^{(2)} \partial_\mu \phi_a(x) \partial_\nu \phi_b(y) \partial_\rho \phi_c(z) \partial_\sigma \phi_d(w) \\ &+ \frac{2}{v_D^3} \int \Gamma_{J_{a\mu}(x)J_{b\nu}(y)K_0(z)}^{(2)} \partial_\mu \phi_a(x) \partial_\nu \phi_b(y) (\phi_c \square \phi_c)(z) \\ &+ \frac{1}{2v_D^2} \int \Gamma_{K_0(x)K_0(y)}^{(2)} (\phi_a \square \phi_a)(x) (\phi_b \square \phi_b)(y). \end{aligned} \quad (6.15)$$

This formula exhibits a functional dependence of  $\Gamma_{\phi_{a_1}\phi_{a_2}\phi_{a_3}\phi_{a_4}}^{(2)}$  on the ancestor amplitudes as in the one loop approximation (see eq. (4.19)). This is a rather surprising result which holds as a consequence of the transversality of the one-loop  $JJ$  ancestor amplitude.

## 7. Hierarchy and Finite Renormalizations

From the results of sections 3 and 5 it is clear that for any solution of the local functional equation (2.5) the knowledge of the ancestor amplitudes order by order in the loop expansion completely determines the dependence on the pion fields. One important consequence of this result is that it has been obtained without relying on the specific subtraction procedure. In particular if we want to perform any subtraction in order to define the theory in  $D = 4$ , it is sufficient to operate on the ancestor amplitudes. The subtractions on the amplitudes involving any number of pions are induced by the integration of the functional equation which has been developed in the previous sections.

In this section we exploit this property in order to shed light on the finite renormalizations allowed from a mathematical point of view by the local symmetry and the weak power-counting theorem.

For that purpose we remark that a sufficient condition for the fulfillment of the local functional equation (2.5) is conjectured to be (in the presence of a symmetric regularization like Dimensional Regularization [3]) the validity of the same functional equation (2.5) for the functional

$$\widehat{\Gamma} = \Gamma^{(0)} + \sum_{k=1}^{\infty} \widehat{\Gamma}^{(k)} \tag{7.1}$$

where  $\Gamma^{(0)}$  is the classical action in eq. (2.3) (giving rise to the tree-level Feynman rules) while  $\widehat{\Gamma}^{(k)}$  collects the  $k$ -th order counterterms. From the mathematical point of view the latter may contain  $k$ -th order finite renormalizations compatible with the symmetry properties and the weak power-counting bounds [2]. This conjecture is supported by formal arguments [18] and by some explicit two-loop examples [3].

We will now prove that the ancestor amplitudes of  $\widehat{\Gamma}$  can be obtained from the tree-level ancestor amplitudes through a suitable redefinition of the classical sources  $J_{a\mu}$  and  $K_0$ :

$$\begin{aligned} J_{a\mu} &\rightarrow J_{a\mu} + A_{1,a\mu}(J) + A_{2,a\mu}(J) + \dots, \\ K_0 &\rightarrow K_0(1 + B_1(K_0, J) + B_2(K_0, J) + \dots) \end{aligned} \tag{7.2}$$

where  $A_{j,a\mu}, B_j$  are of order  $\hbar^j$ .  $A_{j,a\mu}$  does not depend on  $K_0$ . We also set  $A_{0,a\mu} = J_{a\mu}, B_0 = 1$ .

First we notice that by using integration by parts it is always possible to decompose in a unique way an integrated local functional  $\int d^D x X(J, K_0)$  according to

$$\int d^D x X(J, K_0) = \int d^D x \left( J_{a\mu} \mathcal{P}_a^\mu[X] + K_0 \mathcal{Q}[X] \right) \tag{7.3}$$

where  $\mathcal{P}_a^\mu[X]$  is the result of the projection of  $X$  into a local function of  $J$  and its derivatives while  $\mathcal{Q}[X]$  includes also local dependence on  $K_0$  and its derivatives. In order to determine the unknown functions  $A_{j,a\mu}$  and  $B_j$  in eq. (7.2) we perform the substitution (7.2) into

$$\Gamma^{(0)}[0, J_{a\mu}, K_0] = \int d^D x \left( \frac{v_D^2}{8} J^2 + v_D K_0 \right)$$

$$\rightarrow \sum_{l=0}^{\infty} \int d^D x \left( \frac{v_D^2}{8} \sum_{j=0}^l A_{j,a\mu} A_{l-j,a}^\mu + v_D K_0 B_l \right) \quad (7.4)$$

and then compare the second line of the above equation with the ancestor counterterms

$$\widehat{\Gamma}^{(l)}[0, K_0, J_{a\mu}] \equiv \int d^D x \widehat{\mathcal{L}}_l(J, K_0).$$

This gives

$$\begin{aligned} \int d^D x \widehat{\mathcal{L}}_l &= \frac{v_D^2}{8} \int d^D x \sum_{j=0}^l A_{j,a\mu} A_{l-j,a}^\mu + \int d^D x v_D K_0 B_l \\ &= \int d^D x \left( \frac{v_D^2}{4} J_{a\mu} A_{l,a}^\mu + \frac{v_D^2}{8} \sum_{j=1}^{l-1} A_{j,a\mu} A_{l-j,a}^\mu + v_D K_0 B_l \right), \quad l = 1, 2, 3, \dots \end{aligned} \quad (7.5)$$

and hence we derive the recursive solution

$$\begin{aligned} B_0 &= 1, & B_l &= \frac{1}{v_D} \mathcal{Q}[\widehat{\mathcal{L}}_l], \\ A_{0,a\mu} &= J_{a\mu}, \\ A_{l,a\mu} &= \frac{4}{v_D^2} \mathcal{P}_{a\mu}[\widehat{\mathcal{L}}_l] - \frac{1}{2} \mathcal{P}_{a\mu} \left[ \sum_{j=1}^{l-1} A_{j,b\nu} A_{l-j,b}^\nu \right], \quad l = 1, 2, 3, \dots \end{aligned} \quad (7.6)$$

This result states that all possible finite renormalizations in  $\widehat{\Gamma}^{(k)}$ ,  $k > 1$ , compatible with the local symmetry and the weak power-counting, can in fact be interpreted as a redefinition of the sources  $J_{a\mu}$  and  $K_0$  by finite quantum corrections. The latter correspond to the ambiguities allowed in the effective field theory approach discussed in [18].

## 8. Conclusions

The requirement of the invariance of the group Haar measure under local left multiplication can be implemented by a local functional equation for the 1-PI vertex functional of the nonlinear sigma model. This equation can be preserved by the subtraction procedure and completely fixes the dependence of the vertex functional on the pion fields in terms of the ancestor amplitudes (i.e. amplitudes only involving the flat connection and the nonlinear sigma model constraint).

Very remarkably the recursive solution can be written in a very compact form in terms of invariant variables (inducing an implicit dependence of the vertex functional on the quantized field) plus (at order  $n \geq 2$ ) a contribution yielding an explicit dependence on  $\phi_a$ . The latter is fixed by lower order terms (see eq. (5.12)) and does not affect the  $n$ -th loop ancestor amplitudes. This solution provides the full dependence of the 1-PI symmetric amplitudes on the pion fields.

From a technical point of view the method which has been developed in order to integrate the local functional equation extends the cohomological techniques originally developed in the context of gauge theories. In particular it deals with the full Green



functions of the theory (no locality restrictions) and it solves explicitly the inhomogeneous equation (arising from the loop expansion of the bilinear local functional equation) in the absence of multiplicative renormalization (as it happens for the subtraction procedure of the nonlinear sigma model).

The integration of the local functional equation at higher orders in the loop expansion allows to treat a new class of problems which could not be addressed by the knowledge of the solutions of the linearized functional equation only.

Among them we think that two issues are worthwhile to be pointed out. The first one is that our method allows the determination of all pion amplitudes at higher orders in Chiral Perturbation Theory.

The second one is the possibility to investigate the use of the techniques discussed in this paper in order to set up a consistent framework for the study of the structure of the higher order divergences within the program of the quantization of the Stückelberg model for non-abelian massive gauge bosons.

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## A. Consistency condition

In this appendix we verify eq. (5.7) as a consequence of the recursive validity of the functional equation at lower orders. The technique is a variant of the general proof of the consistency condition in the Batalin-Vilkovisky (BV) formalism [17]. One should notice that in the present case the introduction of the antifield  $J_{a\mu}^*$  for the background source  $J_{a\mu}$  is forbidden (since this would lead to an empty cohomology [19]). Therefore one cannot use the standard BV bracket.

The local functional equation at order  $n$  in the loop expansion reads

$$s\Gamma^{(n)} = -\frac{1}{2} \sum_{j=1}^{n-1} \int d^D x \omega_a \frac{\delta\Gamma^{(j)}}{\delta K_0} \frac{\delta\Gamma^{(n-j)}}{\delta\phi_a}, \tag{A.1}$$

which is useful to rewrite in the more symmetric form

$$s\Gamma^{(n)} = -\frac{1}{2} \sum_{j=1}^{n-1} \langle \Gamma^{(j)}, \Gamma^{(n-j)} \rangle. \tag{A.2}$$

In the above equation we have adopted the notation

$$\langle X, Y \rangle = \int d^D x \frac{1}{2} \omega_a \left( \frac{\delta X}{\delta K_0} \frac{\delta Y}{\delta\phi_a} + \frac{\delta Y}{\delta K_0} \frac{\delta X}{\delta\phi_a} \right). \tag{A.3}$$

The following properties hold for  $\langle X, Y \rangle$ :

$$\begin{aligned} \langle X, Y \rangle &= \langle Y, X \rangle, \\ s\langle X, Y \rangle &= -\langle sX, Y \rangle - \langle X, sY \rangle \quad X, Y \text{ bosonic}. \end{aligned} \tag{A.4}$$

We denote by  $\Delta^{(n)}$  the r.h.s. of eq. (A.2), i.e. we set

$$\Delta^{(n)} = -\frac{1}{2} \sum_{j=1}^{n-1} \langle \Gamma^{(j)}, \Gamma^{(n-j)} \rangle. \tag{A.5}$$

If a solution to eq. (A.2) exists, by the nilpotency of  $s$  the following consistency condition has to be verified:

$$s\Delta^{(n)} = 0. \tag{A.6}$$

Let us verify that this is indeed the case under the recursive assumption that the master equation has been fulfilled up to order  $n - 1$ .

By using eq. (A.4) we get

$$\begin{aligned} s\Delta^{(n)} &= s\left(-\frac{1}{2} \sum_{j=1}^{n-1} \langle \Gamma^{(j)}, \Gamma^{(n-j)} \rangle\right) \\ &= +\frac{1}{2} \sum_{j=1}^{n-1} \left( \langle s\Gamma^{(j)}, \Gamma^{(n-j)} \rangle + \langle \Gamma^{(j)}, s\Gamma^{(n-j)} \rangle \right) \\ &= +\frac{1}{2} \sum_{j=1}^{n-1} \left( \langle s\Gamma^{(j)}, \Gamma^{(n-j)} \rangle + \langle s\Gamma^{(n-j)}, \Gamma^{(j)} \rangle \right) \\ &= \sum_{j=1}^{n-1} \langle s\Gamma^{(j)}, \Gamma^{(n-j)} \rangle \end{aligned} \tag{A.7}$$

Now we use the recursive assumption that

$$s\Gamma^{(j)} = -\frac{1}{2} \sum_{k=1}^{j-1} \langle \Gamma^{(k)}, \Gamma^{(j-k)} \rangle \tag{A.8}$$

so that

$$\begin{aligned} s\Delta^{(n)} &= -\frac{1}{2} \sum_{j=1}^{n-1} \sum_{k=1}^{j-1} \langle \langle \Gamma^{(k)}, \Gamma^{(j-k)} \rangle, \Gamma^{(n-j)} \rangle \\ &= -\frac{1}{2} \cdot \frac{1}{3} \sum_{j=1}^{n-1} \sum_{k=1}^{j-1} \left( \langle \langle \Gamma^{(k)}, \Gamma^{(j-k)} \rangle, \Gamma^{(n-j)} \rangle + \langle \langle \Gamma^{(j-k)}, \Gamma^{(n-j)} \rangle, \Gamma^{(k)} \rangle \right. \\ &\quad \left. + \langle \langle \Gamma^{(n-j)}, \Gamma^{(k)} \rangle, \Gamma^{(j-k)} \rangle \right). \end{aligned} \tag{A.9}$$

It turns out that the symmetrized bracket enjoys the following Jacobi identity ( $X, Y, Z$  are assumed to be bosonic):

$$\langle \langle X, Y \rangle, Z \rangle + \langle \langle Z, X \rangle, Y \rangle + \langle \langle Y, Z \rangle, X \rangle = 0. \tag{A.10}$$

The proof of the above equation is provided in the next subsection. By using eq. (A.10) into eq. (A.9) we finally get

$$s\Delta^{(n)} = 0. \quad (\text{A.11})$$

### A.1 Proof of the Jacobi identity for the symmetrized bracket

We assume  $X, Y, Z$  to be bosonic. We write explicitly  $\langle\langle X, Y \rangle, Z \rangle$ :

$$\begin{aligned} \langle\langle X, Y \rangle, Z \rangle &= \int d^D x \frac{1}{2} \omega_a(x) \frac{\delta}{\delta K_0(x)} (\langle X, Y \rangle) \frac{\delta Z}{\delta \phi_a(x)} \\ &\quad + \int d^D x \frac{1}{2} \omega_a(x) \frac{\delta Z}{\delta K_0(x)} \frac{\delta}{\delta \phi_a(x)} (\langle X, Y \rangle) \\ &= \int d^D x \frac{1}{2} \omega_a(x) \frac{\delta}{\delta K_0(x)} \left[ \int d^D y \frac{1}{2} \omega_b(y) \frac{\delta X}{\delta K_0(y)} \frac{\delta Y}{\delta \phi_b(y)} \right. \\ &\quad \left. + \int d^D y \frac{1}{2} \omega_b(y) \frac{\delta Y}{\delta K_0(y)} \frac{\delta X}{\delta \phi_b(y)} \right] \frac{\delta Z}{\delta \phi_a(x)} \\ &\quad + \int d^D x \frac{1}{2} \omega_a(x) \frac{\delta Z}{\delta K_0(x)} \frac{\delta}{\delta \phi_a(x)} \left[ \int d^D y \frac{1}{2} \omega_b(y) \frac{\delta X}{\delta K_0(y)} \frac{\delta Y}{\delta \phi_b(y)} \right. \\ &\quad \left. + \int d^D y \frac{1}{2} \omega_b(y) \frac{\delta Y}{\delta K_0(y)} \frac{\delta X}{\delta \phi_b(y)} \right] \end{aligned} \quad (\text{A.12})$$

We notice that the following terms in the r.h.s. of eq. (A.12)

$$\begin{aligned} &\int d^D x d^D y \frac{1}{2} \omega_a(x) \frac{1}{2} \omega_b(y) \frac{\delta Z}{\delta K_0(x)} \frac{\delta X}{\delta K_0(y)} \frac{\delta^2 Y}{\delta \phi_a(x) \delta \phi_b(y)}, \\ &\int d^D x d^D y \frac{1}{2} \omega_a(x) \frac{1}{2} \omega_b(y) \frac{\delta Z}{\delta K_0(x)} \frac{\delta Y}{\delta K_0(y)} \frac{\delta^2 X}{\delta \phi_a(x) \delta \phi_b(y)} \end{aligned} \quad (\text{A.13})$$

are zero since  $\omega_a(x)$  and  $\omega_b(y)$  are anticommuting.

We make use of eq. (A.12) in order to write the sum  $\langle\langle X, Y \rangle, Z \rangle + \text{cyclic}$ . We organize the terms according to the number of derivatives w.r.t  $K_0$  acting on a single functional. We obtain

$$\begin{aligned} \langle\langle X, Y \rangle, Z \rangle + \text{cyclic} &= \int d^D x \int d^D y \frac{1}{2} \omega_a(x) \frac{1}{2} \omega_b(y) \\ &\quad \times \left[ \frac{\delta^2 X}{\delta K_0(x) \delta K_0(y)} \left( \frac{\delta Y}{\delta \phi_b(y)} \frac{\delta Z}{\delta \phi_a(x)} + \frac{\delta Z}{\delta \phi_b(y)} \frac{\delta Y}{\delta \phi_a(x)} \right) \right. \\ &\quad + \frac{\delta^2 Y}{\delta K_0(x) \delta K_0(y)} \left( \frac{\delta Z}{\delta \phi_b(y)} \frac{\delta X}{\delta \phi_a(x)} + \frac{\delta X}{\delta \phi_b(y)} \frac{\delta Z}{\delta \phi_a(x)} \right) \\ &\quad \left. + \frac{\delta^2 Z}{\delta K_0(x) \delta K_0(y)} \left( \frac{\delta X}{\delta \phi_b(y)} \frac{\delta Y}{\delta \phi_a(x)} + \frac{\delta Y}{\delta \phi_b(y)} \frac{\delta X}{\delta \phi_a(x)} \right) \right] \\ &\quad + \int d^D x \int d^D y \frac{1}{2} \omega_a(x) \frac{1}{2} \omega_b(y) \\ &\quad \times \left[ \frac{\delta X}{\delta K_0(y)} \left( \frac{\delta^2 Y}{\delta K_0(x) \delta \phi_b(y)} \frac{\delta Z}{\delta \phi_a(x)} + \frac{\delta^2 Z}{\delta K_0(x) \delta \phi_b(y)} \frac{\delta Y}{\delta \phi_a(x)} \right) \right. \\ &\quad + \frac{\delta X}{\delta K_0(x)} \left( \frac{\delta^2 Y}{\delta \phi_a(x) \delta K_0(y)} \frac{\delta Z}{\delta \phi_b(y)} + \frac{\delta^2 Z}{\delta \phi_a(x) \delta K_0(y)} \frac{\delta Y}{\delta \phi_b(y)} \right) \\ &\quad \left. + \text{cyclic} \right] \end{aligned} \quad (\text{A.14})$$

The terms in the first block between square brackets in the above equation vanish by symmetry once the anticommutativity of  $\omega_a(x), \omega_b(y)$  is taken into account.

The second block requires some manipulations. If one exchanges  $y \leftrightarrow x$  and  $a \leftrightarrow b$  in the second line of the second block, the latter becomes

$$\begin{aligned}
 & + \int d^D x \int d^D y \frac{1}{2} \omega_a(x) \frac{1}{2} \omega_b(y) \\
 & \quad \times \frac{\delta X}{\delta K_0(y)} \left( \frac{\delta^2 Y}{\delta K_0(x) \delta \phi_b(y)} \frac{\delta Z}{\delta \phi_a(x)} + \frac{\delta^2 Z}{\delta K_0(x) \delta \phi_b(y)} \frac{\delta Y}{\delta \phi_a(x)} \right) \\
 & + \int d^D x \int d^D y \frac{1}{2} \omega_b(y) \frac{1}{2} \omega_a(x) \\
 & \quad \times \frac{\delta X}{\delta K_0(y)} \left( \frac{\delta^2 Y}{\delta K_0(x) \delta \phi_b(y)} \frac{\delta Z}{\delta \phi_a(x)} + \frac{\delta^2 Z}{\delta K_0(x) \delta \phi_b(y)} \frac{\delta Y}{\delta \phi_a(x)} \right) \\
 & \quad + \text{cyclic}
 \end{aligned} \tag{A.15}$$

The above expression is zero since  $\omega_a(x), \omega_b(y)$  anticommute.

Therefore we establish the Jacobi identity for the symmetrized bracket in the form

$$\langle\langle X, Y \rangle, Z \rangle + \langle\langle Z, X \rangle, Y \rangle + \langle\langle Y, Z \rangle, X \rangle = 0 \tag{A.16}$$

with  $X, Y, Z$  bosonic.

## B. Integrability condition

In this appendix we check that eq. (5.11) holds as a consequence of eq. (5.7). Eq. (5.7) reads in the variables  $\{\bar{K}_0, j_{a\mu}, \phi_a, \bar{\omega}_a\}$

$$\int d^D x \bar{\omega}_a(x) \frac{\delta \Delta^{(n)}}{\delta \phi_a(x)} = 0. \tag{B.1}$$

By differentiating the above equation w.r.t.  $\bar{\omega}_a(x), \bar{\omega}_b(y)$  and by setting  $\bar{\omega} = 0$  we get

$$\frac{\delta^2 \Delta^{(n)}}{\delta \bar{\omega}_b(y) \delta \phi_a(x)} = \frac{\delta^2 \Delta^{(n)}}{\delta \bar{\omega}_a(x) \delta \phi_b(y)}. \tag{B.2}$$

Let us now consider the r.h.s. of eq. (5.11). For each permutation  $\sigma \in S_m$  there exists a unique integer  $1 \leq K \leq m$  such that  $\sigma(K) = 1$ . Therefore (we drop here the dependence on  $\zeta_1, \dots, \zeta_n$  since the latter does not play any role in the following argument)

$$\begin{aligned}
 \frac{1}{m!} \sum_{\sigma \in S_m} \Delta_{\bar{\omega}_{\sigma(1)} \phi_{\sigma(2)} \dots \phi_{\sigma(m)}}^{(n)} &= \frac{1}{m!} \sum_{\sigma \in S_{m-1}[2, \dots, m]} \Delta_{\bar{\omega}_{a_1} \phi_{\sigma(2)} \dots \phi_{\sigma(m)}}^{(n)} \\
 &+ \frac{1}{m!} \sum_{K=2}^m \sum_{\sigma \in S_{m-1}[1, 2, \dots, \hat{K}, \dots, m]} \Delta_{\bar{\omega}_{\sigma(1)} \phi_{\sigma(K)} \phi_{\sigma(2)} \dots \hat{\phi}_{\sigma(K)} \dots \phi_{\sigma(m)}}^{(n)}.
 \end{aligned} \tag{B.3}$$

In the above equation a hat over a variable denotes omission of the latter from the relevant list and  $S_{m-1}[a, b, \dots, c]$  denotes the group of permutations over the  $m - 1$  elements  $\{a, b, \dots, c\}$ .

We now use eq. (B.2) in the second line of eq. (B.3) as well as the fact that  $\sigma(K) = 1$  and we get

$$\begin{aligned} \frac{1}{m!} \sum_{\sigma \in S_m} \Delta_{\bar{\omega}_{a_{\sigma(1)} \phi_{a_{\sigma(2)} \dots \phi_{a_{\sigma(m)}}}}^{(n)} &= \frac{1}{m!} \sum_{\sigma \in S_{m-1}[2, \dots, m]} \Delta_{\bar{\omega}_{a_1 \phi_{a_{\sigma(2)} \dots \phi_{a_{\sigma(m)}}}}^{(n)} \\ &+ \frac{1}{m!} \sum_{K=2}^m \sum_{\sigma \in S_{m-1}[1, 2, \dots, \hat{K}, \dots, m]} \Delta_{\bar{\omega}_{a_1 \phi_{a_{\sigma(1)} \phi_{a_{\sigma(2)} \dots \hat{\phi}_{a_{\sigma(K)} \dots \phi_{a_{\sigma(m)}}}}}}^{(n)}. \end{aligned} \tag{B.4}$$

By the Bose statistics of the  $\phi$ 's we also get

$$\begin{aligned} \frac{1}{m!} \sum_{\sigma \in S_m} \Delta_{\bar{\omega}_{a_{\sigma(1)} \phi_{a_{\sigma(2)} \dots \phi_{a_{\sigma(m)}}}}^{(n)} &= \frac{1}{m} \Delta_{\bar{\omega}_{a_1 \phi_{a_2} \dots \phi_{a_m}}^{(n)} + \frac{m-1}{m} \Delta_{\bar{\omega}_{a_1 \phi_{a_2} \phi_{a_3} \dots \phi_{a_m}}^{(n)} \\ &= \Delta_{\bar{\omega}_{a_1 \phi_{a_2} \dots \phi_{a_m}}^{(n)}, \end{aligned} \tag{B.5}$$

which proves eq. (5.11).

A comment is in order here. It is a well-known fact in cohomological algebra [14–16] that if a local functional with ghost number one satisfies the consistency condition in eq. (5.7) (i.e. it is BRST closed) and the BRST differential  $s$  has been trivialized by reduction to a doublet pair

$$s\phi_a = \bar{\omega}_a, \quad s\bar{\omega}_a = 0$$

then that functional is also BRST-exact.

The present analysis generalizes this result to the case of arbitrary functionals, the locality property being nowhere used in the above construction.

### C. One-loop invariants

We report here the invariants parameterizing the one-loop divergences of the nonlinear sigma model in  $D = 4$  [2]. The background connection is denoted by  $J_{a\mu}$ .

$$\begin{aligned} \mathcal{I}_1 &= \int d^D x \left[ D_\mu(F - J)_\nu \right]_a \left[ D^\mu(F - J)^\nu \right]_a, \\ \mathcal{I}_2 &= \int d^D x \left[ D_\mu(F - J)^\mu \right]_a \left[ D_\nu(F - J)^\nu \right]_a, \\ \mathcal{I}_3 &= \int d^D x \epsilon_{abc} \left[ D_\mu(F - J)_\nu \right]_a \left( F_b^\mu - J_b^\mu \right) \left( F_c^\nu - J_c^\nu \right), \\ \mathcal{I}_4 &= \int d^D x \left( \frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a} \right)^2, \\ \mathcal{I}_5 &= \int d^D x \left( \frac{m_D^2 K_0}{\phi_0} - \phi_a \frac{\delta S_0}{\delta \phi_a} \right) \left( F_b^\mu - J_b^\mu \right)^2, \\ \mathcal{I}_6 &= \int d^D x \left( F_a^\mu - J_a^\mu \right)^2 \left( F_b^\nu - J_b^\nu \right)^2, \\ \mathcal{I}_7 &= \int d^D x \left( F_a^\mu - J_a^\mu \right) \left( F_a^\nu - J_a^\nu \right) \left( F_{b\mu} - J_{b\mu} \right) \left( F_{b\nu} - J_{b\nu} \right). \end{aligned} \tag{C.1}$$

In the above equation  $D_\mu[F]$  stands for the covariant derivative w.r.t.  $F_{a\mu}$

$$D_\mu[F]_{ab} = \delta_{ab} \partial_\mu + \epsilon_{acb} F_{c\mu}. \tag{C.2}$$

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